Energy in a standing wave

- Previously, we saw for traveling waves that energy is transported in the direction of wave propagation at the wave velocity v
- B/c standing waves comprised of two equal and oppositely traveling waves, the transported energy from each of these traveling wave components exactly cancels, so there is no net energy transport
- However, there still is obviously energy in a standing wave, both potential and kinetic, it is just not "moving" in space
- Consider the taut string example. We previously derived the energy contained in a portion $a \leq x \leq b$ undergoing transverse oscillations:

$$E = \frac{1}{2}\mu \int_a^b dx \left[\underbrace{\left(\frac{\partial y}{\partial t}\right)^2}_{\text{kinetic energy}} + \underbrace{v^2 \left(\frac{\partial y}{\partial x}\right)^2}_{\text{potential energy}} \right]$$

μ: mass per unit length

- We now use this expression to get total energy of a standing wave of string of length L vibrating in a single mode n

solutions: $y_n(x,t) = A_n \sin\left(\frac{n\pi}{L} x\right) \cos \omega_n t$

[derived last lecture] where $\omega_n = v \frac{n\pi}{L}$

- Differentiate to get $\frac{\partial y}{\partial t}$ & $\frac{\partial y}{\partial x}$

$$\Rightarrow \frac{\partial y_n}{\partial t} = -A_n \omega_n \sin\left(\frac{n\pi}{L}x\right)\sin\omega_n t$$

$$\frac{\partial y_n}{\partial x} = A_n\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi}{L}x\right)\cos\omega_n t$$

- Substitute these into expression for energy, integrating from $0 \to L$

$$E_n = \frac{1}{2}\mu A_n^2\left[\omega_n^2 \sin^2\omega_n t \boxed{\int_0^L dx \sin^2\left(\frac{n\pi}{L}x\right)} + v^2\left(\frac{n\pi}{L}\right)^2\cos^2\omega_n t \boxed{\int_0^L dx \cos^2\left(\frac{n\pi}{L}x\right)}\right]$$

- The two integrals above in the $\boxed{}$ boxes integrate to the same value $\boxed{} = \frac{L}{2}$ (see book for derivation)
- Substitute this into E_n, and use $\omega_n = v\left(\frac{n\pi}{L}\right)$

$$\Rightarrow E_n = \frac{1}{4}\mu L A_n^2 \omega_n^2 \underbrace{(\sin^2\omega_n t + \cos^2\omega_n t)}_{=1}$$

$$= \frac{1}{4}\mu L A_n^2 \omega_n^2$$

From kinetic energy

From potential energy

$\rightarrow$ Energy flows b/t kinetic & potential, but total is conserved

$\rightarrow$ similar to simple harmonic oscillator!

Standing waves as normal modes of a vibrating string

- Several lectures ago, we discussed "normal modes" of a coupled harmonic oscillator. ~~Few~~ Some observations:

  → All masses oscillate at same frequency

  → normal modes independent of each other [recall the "q_1" and "q_2" coordinate transform discussion]

  → general solution is a superposition of normal modes

→ All this also applies to standing waves! Each mode is given by different value of n in $y_n(x,t)$

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Superposition principle

- If $y_1(x,t)$ and $y_2(x,t)$ are any two solutions of the wave eq., then so is any linear combination

$$y(x,t) = A_1\, y_1(x,t) + A_2\, y_2(x,t)$$

where A_1, A_2 are arbitrary constants

[see book for proof]

Superposition of normal modes

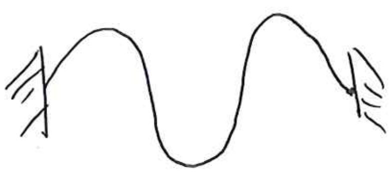
- Apply idea of superposition principle to normal modes on a vibrating string:

$$y_n(x,t) = A_n \sin\left(\frac{n\pi}{L}x\right)\cos\omega_n t \qquad \omega_n = \frac{n\pi v}{L}$$

- In general, motion of the string given by superposition of these normal modes:

$$y(x,t) = \sum_n y_n(x,t) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right)\cos\omega_n t$$

- Example:

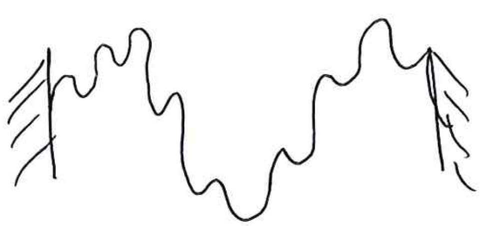


$n=3$ mode
$A_3 = 1$

$+$

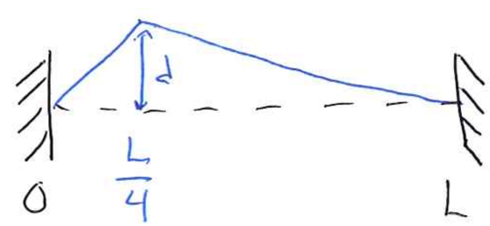
$n=9$ mode
$A_9 = 0.5$

$=$



(I drew too many anti-nodes for $n=9$, but you get the idea...)

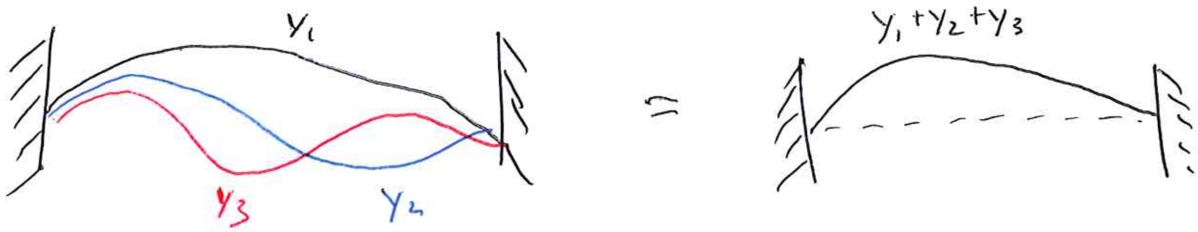
· What happens when we pluck a guiter string?



· At $t=0$, we displace string by distance d at location $L/4$ by plucking it

· Clearly this shape does not correspond to any of the normal modes

· The triangular shape can be recreated by superposition of normal modes!:

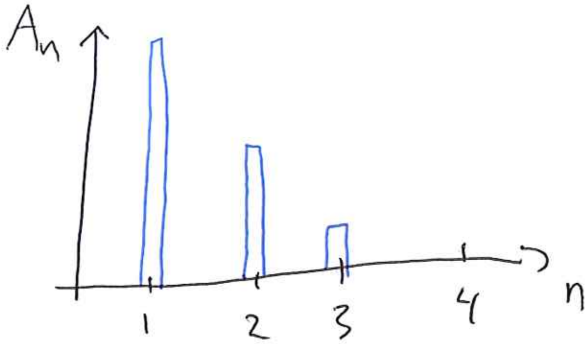


$$\Rightarrow y(x,0) = y_1(x,0) + y_2(x,0) + y_3(x,0) + \dots$$

· Each mode oscillates w/ frequency $\omega_n = \frac{n\pi v}{L}$ after release $(t>0)$

· The amplitude A_n of each excited normal mode depends on the initial condition (ie, what best reproduces the shape at $t=0$ when being plucked)

· Represent these amplitudes in "Frequency spectrum"

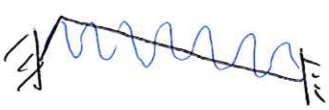


· Why no $n=4$?: B/c we plucked at $\frac{L}{4}$ where there is a node $(y=0)$

→ so not excited in this example!

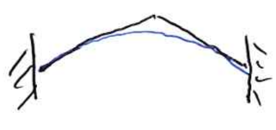
- The above example shows we can adjust (to some extent) what modes are excited depending where we pluck the string.

→ pluck close to a boundary,



→ need higher n's to reproduce the short wavelength components required for the steep rise → "bright" sound

- pluck in the middle



→ primarily excite fundamental → "mellow"/dark sound

- Different combinations of harmonics called "timbre" (in music) not physics

→ controlled in where/how pluck string (or play ~~note~~ sound in other instrument types)

- This is why different instruments sound different. They all have the same fundamental pitch when playing a given musical note, but harmonic content is different

→ Why Stradivarious violins worth so much $$$

→ beautiful sounding harmonic content (determined by unique materials/construction process)

Amplitudes of normal modes & Fourier analysis

- In the last example, we saw that initial shape of plucked string $f(x)$ ~~is~~ at $t=0$ is

$$\boxed{y(x,0) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right) = f(x)}$$

- This is a general result, that any shape on the string can be represented by different combinations of normal modes

→ proved by Fourier, called "Fourier series"

- Amplitudes A_n called "Fourier coefficients"

- Basic idea: any function can be represented as sum of different sine and cosine functions via Fourier series

→ Hugely impactful in all areas of physics & engineering!

- Given that $f(x) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right)$, how do we find the various A_n?

- First, note the following two integrals:

$$\int_0^L dx \sin^2\left(\frac{n\pi}{L}x\right) = \frac{L}{2} \qquad (\mathrm{I})$$

$$\int_0^L dx \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right) = 0 \quad m \neq n \qquad (\mathrm{II})$$

- Multiply $f(x)$ by $\sin\left(\frac{m\pi}{L}x\right)$ [on both sides of eq.] and integrate from $x = 0 \to L$:

$$\int_0^L dx \sin\left(\frac{m\pi}{L}x\right) f(x) = \sum_n A_n \underbrace{\int_0^L dx \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)}_{= \begin{cases} L/2 & \text{for } m=n \quad (\mathrm{I}) \\ 0 & \text{for } m \neq n \quad (\mathrm{II}) \end{cases}}$$

$$\Rightarrow \boxed{A_n = \frac{2}{L}\int_0^L dx \sin\left(\frac{n\pi}{L}x\right) f(x)}$$

for $n = 1, 2, \ldots$

[the $\sum_n$ goes away b/c there is only one $A_n \neq 0$ when $m=n$]

- Expressions for $f(x)$ and A_n known as "Fourier theorems"

Energy of vibrating string

- We showed before that the energy of a given normal mode of a vibrating string is:

$$E_n = \frac{1}{4}\mu L A_n^2 \omega_n^2$$

- Using superposition principle:

$$\boxed{E = \sum_n E_n = \frac{1}{4}\mu L \sum_n A_n^2 \omega_n^2}$$

→ sum of energy of all contributing harmonics to vibrations